INFINITESIMAL GENERATOR

A suitable tool for PD estimation

REAL CASE

Eurobanking 2011
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Agenda

- iRating
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- Rating in time
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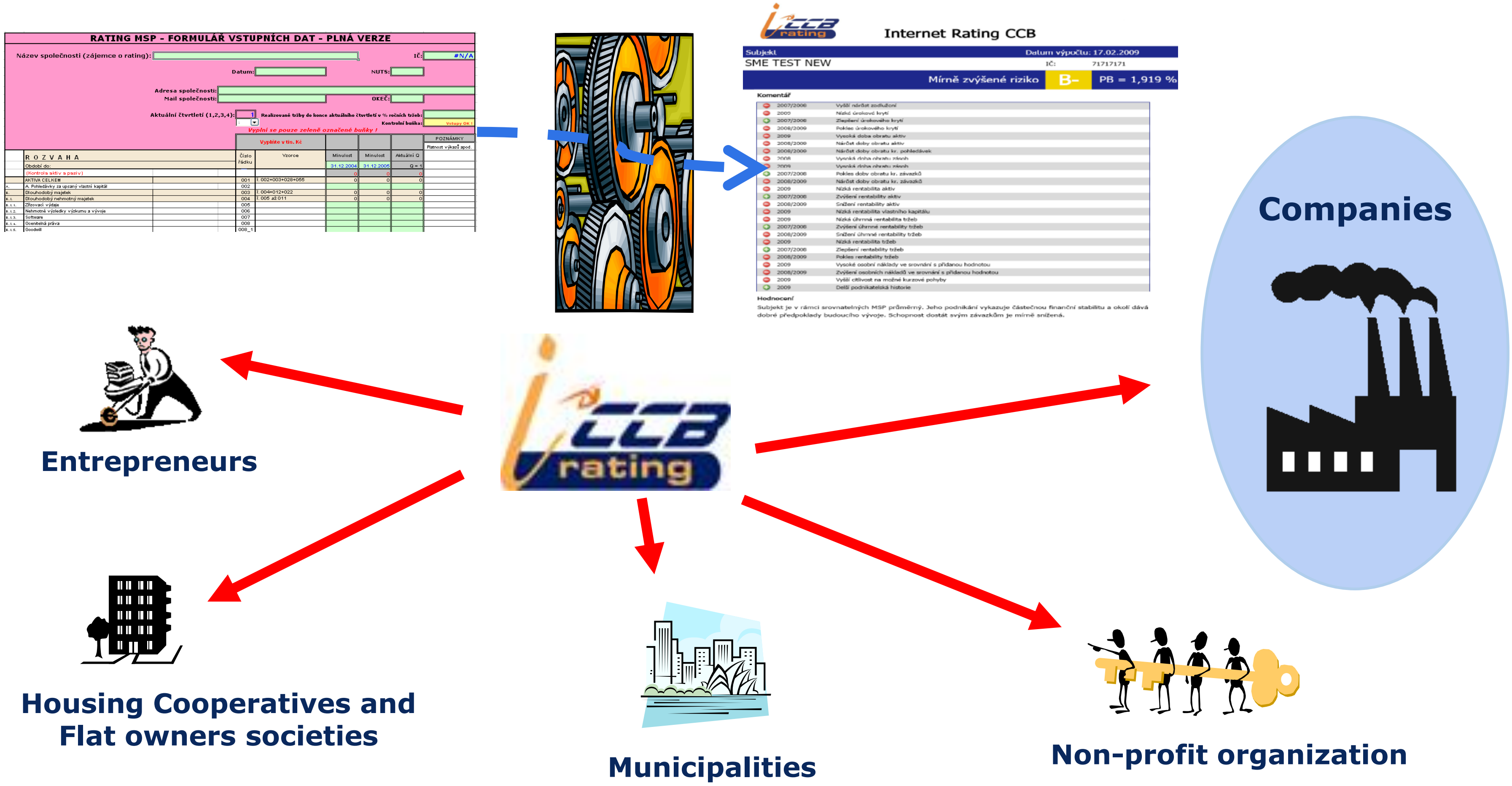
iRating family

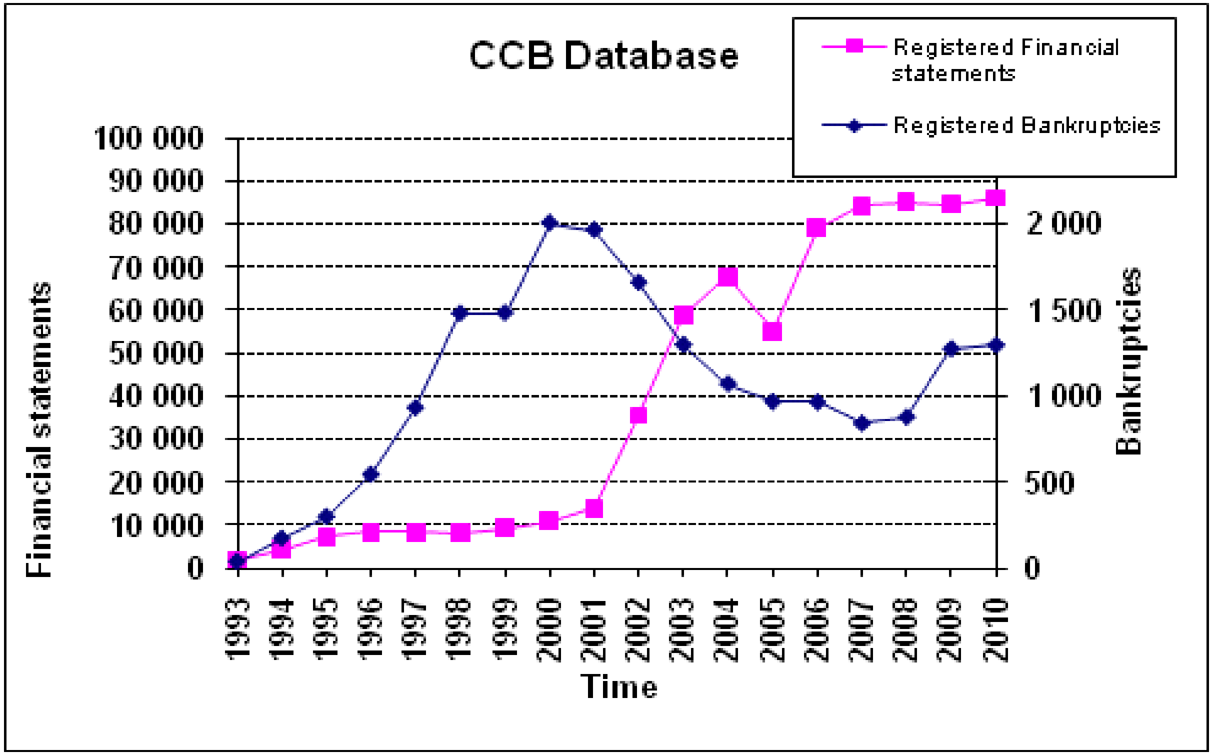
iRating model - tool for exact evaluation various subjects available through web application



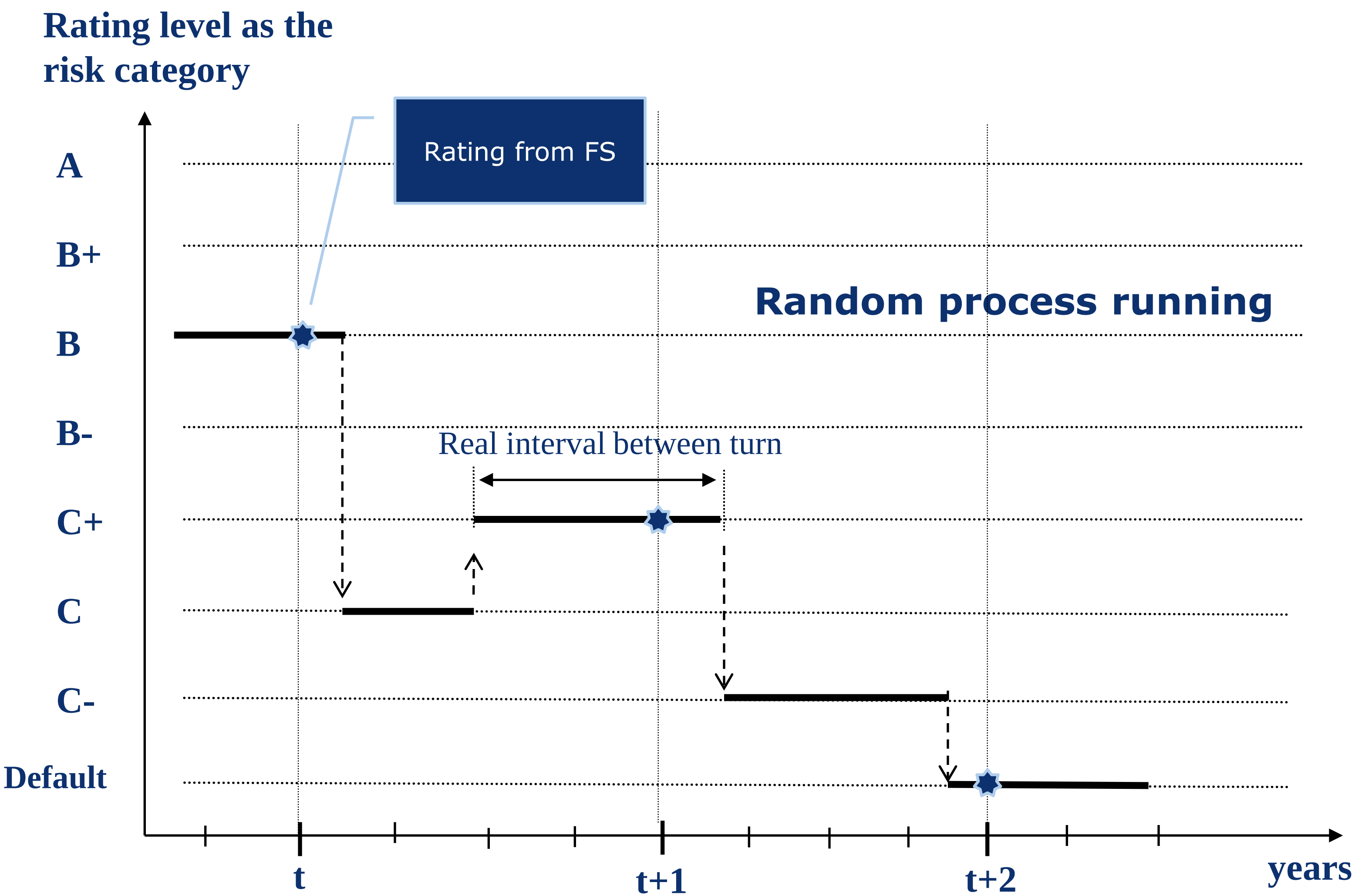
Historical data

Data sources used for calculation

- Commercial Register (Annual reports, **Financial Statements**, ...)
- Bankruptcy Register (**Bankruptcy Date**, Verdict, Court, …)
- Business Register (Identification number, name, region**, industry branch**, ...)



Rating in time



Migration matrix

Measured One-Year Rating Migration Matrix, 1995 to 2009										
Building Industry CZ Revenues from 0,5MM to 1,5MM CZK		Rating To :								Number of Migrations :
		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	**34**	21	9	6	3	2	0	**0**	**75**
	B+	23	**100**	56	21	7	7	0	**0**	**214**
	B	8	64	**126**	56	23	8	3	**0**	**288**
	B-	4	31	74	**145**	90	25	8	**0**	**377**
	C+	3	14	33	132	**216**	68	13	**1**	**480**
	C	2	8	8	28	65	**57**	17	**7**	**192**
	C-	0	0	3	5	10	11	**40**	**13**	**82**
Number of Migrations :		**74**	**238**	**309**	**393**	**414**	**178**	**81**	**21**	**1 708**

Evaluation was done using iRating model for companies with YTO 0,5 to 1,5 MM CZK
Observation period was 15 years (1995 – 2009)
Industry branches are 28, example originate from building industry

Transition matrix

Transition matrix is calculated for each Industry branch

$$\text{Stochastic process } \xi(\mathrm{t}):$$
$$\text{Status space } \mathrm{X} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5, \mathrm{x}_6, \mathrm{x}_7, \mathrm{x}_8\} = \{\mathrm{A}, \mathrm{B+}, \mathrm{B}, \mathrm{B-}, \mathrm{C+}, \mathrm{C}, \mathrm{C-}, \mathrm{Default}\}$$
$$\text{Parametric space } \mathrm{T} = \{\mathrm{t} : 0 \le \mathrm{t} \le +\infty\}$$

Transition matrix is a matrix of the migration probabilities. It is a stochastic matrix as is valid:

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \ldots\ldots & p_{1N}(t) \\ p_{21}(t) & p_{22}(t) & \ldots\ldots & p_{2N}(t) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ p_{N-1,1}(t) & \ldots\ldots & \ldots\ldots & p_{N-1,N}(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 \le p_{ij}(t) \le 1$$
$$\sum_{j=1}^{N} p_{ij}(t) = 1$$

Transition matrix

The estimated transition probability of the migrations from initial rating state **i** to rating state **j** is:

$$p_{ij}(t) = \frac{N_{ij}(t)}{N_i(t)}$$

There were ratings in state **i** at the beginning of the period **t** and ratings migrated from state **i** at the beginning of the period **t** to state **j** at the end of the period .

Transition matrix was estimated as average. It was accomplished by wight- averaging transition matrices over a variety of time horizons. The weights are the numbers of ratings at the beginning of the periods . Average transition matrix is computed by following formula:

$$p_{ij} = \frac{\sum_{t_o} N_i(t_o)\, p_{ij}(t_o)}{\sum_{t_o} N_i(t_o)}$$

Transition matrix

Measured One-Year Rating Migration Matrix, 1995 to 2009										
Building Industry CZ		Rating To :								Number of Migrations :
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	34	21	9	6	3	2	0	0	75
	B+	23	100	56	21	7	7	0	0	214
	B	8	64	126	56	23	8	3	0	288
	B-	4	31	74	145	90	25	8	0	377
	C+	3	14	33	132	216	68	13	1	480
	C	2	8	8	28	65	57	17	7	192
	C-	0	0	3	5	10	11	40	13	82
Number of Migrations :		74	238	309	393	414	178	81	21	1 708

Empirical One-Year Transition Matrix, 1995 to 2009										
Building Industry CZ		Rating To :								Sum of Row
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	45,33%	28,00%	12,00%	8,00%	4,00%	2,67%	0,00%	0,000%	100%
	B+	10,75%	46,73%	26,17%	9,81%	3,27%	3,27%	0,00%	0,000%	100%
	B	2,78%	22,22%	43,75%	19,44%	7,99%	2,78%	1,04%	0,000%	100%
	B-	1,06%	8,22%	19,63%	38,46%	23,87%	6,63%	2,12%	0,000%	100%
	C+	0,63%	2,92%	6,88%	27,50%	45,00%	14,17%	2,71%	0,208%	100%
	C	1,04%	4,17%	4,17%	14,58%	33,85%	29,69%	8,85%	3,646%	100%
	C-	0,00%	0,00%	3,66%	6,10%	12,20%	13,41%	48,78%	15,854%	100%
	Bankruptcy	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100,00%	100%

Infinitesimal generator (matrix A)

Empiric transition matrix presents the probability of migration between risk categories in fix time interval (year). There is obvious disharmony with real risk exposition time which could be arbitrary value (one day, 15 days, 41 days ….)

One of the solution is to utilize infinitesimal generator allowing estimate migration probabilities in free time starting from fix (e.g. yearly)transition matrix.

Infinitesimal generator calculation is based on **Chapman-Kolmogorov formula**, describing schedule of migration probabilities. In that case solution of the CH-K formula has next analytical :

$$P(t) = e^{At} = \sum_{k=1}^{\infty} \frac{A^k}{k!} t^k$$

$$\text{matrix } A \text{ is constant and squere, rank NxN}$$

Infinitesimal generator (matrix A)

The matrix **A** is called infinitesimal generator or generation matrix for Markov process associated with transition matrix **P**. Infinitesimal generator **A** plays a fundamental role in the Markov process framework. Matrix **A** is parametric by time **t**. In case of the homogenous Markov processes matrix **A** is time invariant.

Each entry of the matrix **A** represents the intensity of transition from the rating state **i** to rating state **j**. The infinitesimal generator matrix is also known as the intensity matrix.

$$\sum_{j=1}^{N} a_{ij} = 0, \quad \text{pro } i = 1,..,N \quad \text{(sum of elements in each row is equal zero)}$$

$$a_{ij} \leq 0, \quad \text{pro } i = j \quad \text{(diagonal elements are} \leq 0\text{)}$$

Infinitesimal generator (matrix A)

We are searching generator **A** for measured (estimated) transition matrix **P**, defined for basic time horizon (year, half-year, quarter, month …). There are two ways to accomplish it:

through inverse operator exp(), i.e.:

$$A = \ln(P)$$

or by Taylor expansion:

$$A = \ln(P) \cong (P-I)-(P-I)^2/2+(P-I)^3/3-(P-I)^4/4+\ldots$$

Basic time horizon was chosen as one time horizon, i.e. $\mathbf{t_T} = 1$.

Infinitesimal generator **A** was computed through the use of diagonalizationed decomposition of **P**

$$P = Q \times \Lambda \times Q^{-1}$$

Infinitesimal generator (matrix A)

with eigenvectors as columns of and **Q** eigenvalues on diagonal of **Λ**. Then we can express:

$$A = \ln(P) = \ln(Q \times \Lambda \times Q^{-1}) = Q \times \ln(\Lambda) \times Q^{-1}$$

Correctness of the computed generator A

1. Nonnegative off-diagonal entries. Any negative off-diagonal entries (usually quite small) has to be replaced with 0 and adding their values into the corresponding diagonal entry to preserve the property of having row sums 0 or adding their values into all the entries of the same rows proportional to their absolute values.

2. Rating scale conservation. Lower rating means higher credit risk (higher probability of default) and higher rating means lower credit risk (lower probability of default). This requirement implies following condition on relevant entries of the generator matrix A

$$\sum_{j \geq k} a_{i\,j} \leq \sum_{j \geq k} a_{i+1\,j}$$

For details see

[15] *R.A.Jarrow, D. Lando, and S.M. Turnbull* : A Markov model for the term structure of credit risk spreads. The Review of financial Studies 1997 Vol.10, No.2, pp.481-523

Infinitesimal generator (matrix A)

Infinitesimal Generator A0 (not adjusted negative off-diagonal entries)

Building Industry CZ		Rating To :								
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	Sum of Row
Rating From:	A	**-0,870**	0,631	0,064	0,115	0,033	0,033	-0,005	**0,000**	**0,00**
	B+	0,252	**-1,011**	0,636	0,091	-0,034	0,081	-0,015	**0,000**	**0,00**
	B	-0,002	0,549	**-1,125**	0,507	0,046	0,010	0,015	**-0,001**	**0,00**
	B-	0,004	0,074	0,540	**-1,371**	0,661	0,066	0,031	**-0,005**	**0,00**
	C+	0,006	-0,001	-0,024	0,810	**-1,227**	0,435	0,009	**-0,007**	**0,00**
	C	0,013	0,096	0,012	0,047	1,077	**-1,530**	0,239	**0,045**	**0,00**
	C-	-0,001	-0,038	0,068	0,048	0,113	0,347	**-0,756**	**0,218**	**0,00**
	Bankruptcy	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	**0,00**

Infinitesimal Generator A1 (adjusted negative off-diagonal entries by adding into others entries)

Building Industry CZ		Rating To :								
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	Sum of Raw
Rating From:	A	**-0,873**	0,629	0,064	0,114	0,033	0,033	0,000	**0,000**	**0,00**
	B+	0,246	**-1,035**	0,620	0,089	0,000	0,079	0,000	**0,000**	**0,00**
	B	0,000	0,548	**-1,126**	0,507	0,046	0,010	0,015	**0,000**	**0,00**
	B-	0,004	0,074	0,539	**-1,374**	0,660	0,066	0,031	**0,000**	**0,00**
	C+	0,006	0,000	0,000	0,799	**-1,243**	0,429	0,008	**0,000**	**0,00**
	C	0,013	0,096	0,012	0,047	1,077	**-1,530**	0,239	**0,045**	**0,00**
	C-	0,000	0,000	0,067	0,047	0,110	0,339	**-0,775**	**0,213**	**0,00**
	Bankruptcy	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	**0,00**

Infinitesimal generator (matrix A)

Infinitesimal Generator A2 (adjusted negative off-diagonal entries by adding into diagonal entries)

Building Industry CZ		Rating To :								Sum of Raw
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	-0,875	0,631	0,064	0,115	0,033	0,033	0,000	0,000	0,00
	B+	0,252	-1,060	0,636	0,091	0,000	0,081	0,000	0,000	0,00
	B	0,000	0,549	-1,128	0,507	0,046	0,010	0,015	0,000	0,00
	B-	0,004	0,074	0,540	-1,376	0,661	0,066	0,031	0,000	0,00
	C+	0,006	0,000	0,000	0,810	-1,259	0,435	0,009	0,000	0,00
	C	0,013	0,096	0,012	0,047	1,077	-1,530	0,239	0,045	0,00
	C-	0,000	0,000	0,068	0,048	0,113	0,347	-0,795	0,218	0,00
	Bankruptcy	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,00

Infinitesimal Generator A3 (computed by algorithm JAROW at all)

Building Industry CZ		Rating To :								Sum of Raw
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	-0,791	0,405	0,174	0,116	0,058	0,039	0,000	0,000	0,00
	B+	0,153	-0,761	0,374	0,140	0,047	0,047	0,000	0,000	0,00
	B	0,041	0,327	-0,827	0,286	0,117	0,041	0,015	0,000	0,00
	B-	0,016	0,128	0,305	-0,956	0,371	0,103	0,033	0,000	0,00
	C+	0,009	0,042	0,100	0,399	-0,799	0,206	0,039	0,003	0,00
	C	0,018	0,072	0,072	0,252	0,585	-1,214	0,153	0,063	0,00
	C-	0,000	0,000	0,051	0,085	0,171	0,188	-0,718	0,222	0,00
	Bankruptcy	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,00

Exactness of yearly calculation

Average One-Year Rating Transition Matrix 1995-2009 Computed by Generator A1

Building Industry CZ		Rating To:								Sum of Row
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	**45,05%**	27,29%	11,89%	8,05%	4,40%	2,75%	0,46%	**0,103%**	**100%**
	B+	10,49%	**44,70%**	25,64%	10,11%	4,62%	3,54%	0,74%	**0,166%**	**100%**
	B	2,79%	21,72%	**43,61%**	19,43%	8,23%	2,81%	1,20%	**0,195%**	**100%**
	B-	1,07%	8,15%	19,81%	**38,27%**	23,56%	6,58%	2,13%	**0,421%**	**100%**
	C+	0,65%	3,19%	7,76%	27,23%	**43,82%**	13,96%	2,66%	**0,735%**	**100%**
	C	1,06%	4,34%	4,58%	14,50%	33,37%	**29,61%**	8,70%	**3,835%**	**100%**
	C-	0,26%	1,74%	4,21%	6,14%	11,90%	13,19%	**46,92%**	**15,645%**	**100%**
	Bankruptcy	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	**100,00%**	**100%**

Empirical One-Year Transition Matrix, 1995 to 2009

Building Industry CZ		Rating To :								Sum of Row
Revenues from 0,5MM to 1,5MM CZK		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	**45,33%**	28,00%	12,00%	8,00%	4,00%	2,67%	0,00%	**0,000%**	**100%**
	B+	10,75%	**46,73%**	26,17%	9,81%	3,27%	3,27%	0,00%	**0,000%**	**100%**
	B	2,78%	22,22%	**43,75%**	19,44%	7,99%	2,78%	1,04%	**0,000%**	**100%**
	B-	1,06%	8,22%	19,63%	**38,46%**	23,87%	6,63%	2,12%	**0,000%**	**100%**
	C+	0,63%	2,92%	6,88%	27,50%	**45,00%**	14,17%	2,71%	**0,208%**	**100%**
	C	1,04%	4,17%	4,17%	14,58%	33,85%	**29,69%**	8,85%	**3,646%**	**100%**
	C-	0,00%	0,00%	3,66%	6,10%	12,20%	13,41%	**48,78%**	**15,854%**	**100%**
	Bankruptcy	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	**100,00%**	**100%**

Exactness of yearly calculation

Absolute values of the diferrences between homothetic entries in Empirical Transition Matrix and Transition Matrix Computed by Generator									
	A	**B+**	**B**	**B-**	**C+**	**C**	**C-**	**Bankruptcy**	**Σ Row**
A	0,28%	0,71%	0,11%	0,05%	0,40%	0,09%	0,46%	0,10%	2,20%
B+	0,25%	2,03%	0,53%	0,29%	1,35%	0,27%	0,74%	0,17%	5,62%
B	0,02%	0,51%	0,14%	0,01%	0,25%	0,04%	0,16%	0,20%	1,32%
B-	0,01%	0,07%	0,19%	0,19%	0,31%	0,05%	0,01%	0,42%	1,25%
C+	0,02%	0,27%	0,88%	0,27%	1,18%	0,21%	0,04%	0,53%	3,41%
C	0,02%	0,17%	0,42%	0,08%	0,49%	0,08%	0,16%	0,19%	1,59%
C-	0,26%	1,74%	0,55%	0,04%	0,29%	0,23%	1,86%	0,21%	5,18%
Bankruptcy	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
Σ column	0,87%	5,50%	2,81%	0,93%	4,28%	0,95%	3,43%	1,81%	20,57%

The accuracy of computed transition matrix was evaluated by comparison original transition matrix P and transition matrix generated by generator matrix A. There is used L1-norm of the difference matrix $(P(t)-\exp(At))$, i.e. the sum of the absolute values of the difference matrix.
We compute one year L1-norm that $norm[P(1)-\exp(A\times 1)]$ =0,2057 (relatively 2,94%, experience says more than 5% should be a problem)

Optional transition matrix

Now we have a verified generator A1 so we can calculate transition matrix for any time:

$$\mathbf{P}(t) = \exp(\mathbf{A}\times t) \quad \textit{where}\ \mathbf{A}\ \text{is generator and}\ t\ \text{is specific time}$$

Average 3-Months Rating Transition Matrix 1995-2009 Computed by Generator										
Building Industry CZ Revenues from 0,5MM to 1,5MM CZK		Rating To:								Sum of Row
		A	B+	B	B-	C+	C	C-	Bankruptcy	
Rating From:	A	**80,97%**	12,43%	2,36%	2,49%	0,93%	0,78%	0,04%	**0,005%**	**100%**
	B+	4,93%	**78,24%**	12,22%	2,53%	0,47%	1,52%	0,08%	**0,011%**	**100%**
	B	0,34%	10,50%	**77,18%**	9,51%	1,68%	0,44%	0,34%	**0,012%**	**100%**
	B-	0,14%	2,08%	10,02%	**73,07%**	12,15%	1,83%	0,67%	**0,029%**	**100%**
	C+	0,15%	0,29%	1,02%	14,67%	**75,55%**	7,81%	0,46%	**0,059%**	**100%**
	C	0,31%	1,81%	0,55%	2,80%	19,18%	**69,80%**	4,49%	**1,071%**	**100%**
	C-	0,02%	0,18%	1,43%	1,31%	3,13%	6,60%	**82,40%**	**4,935%**	**100%**
	Bankruptcy	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	**100,00%**	**100%**

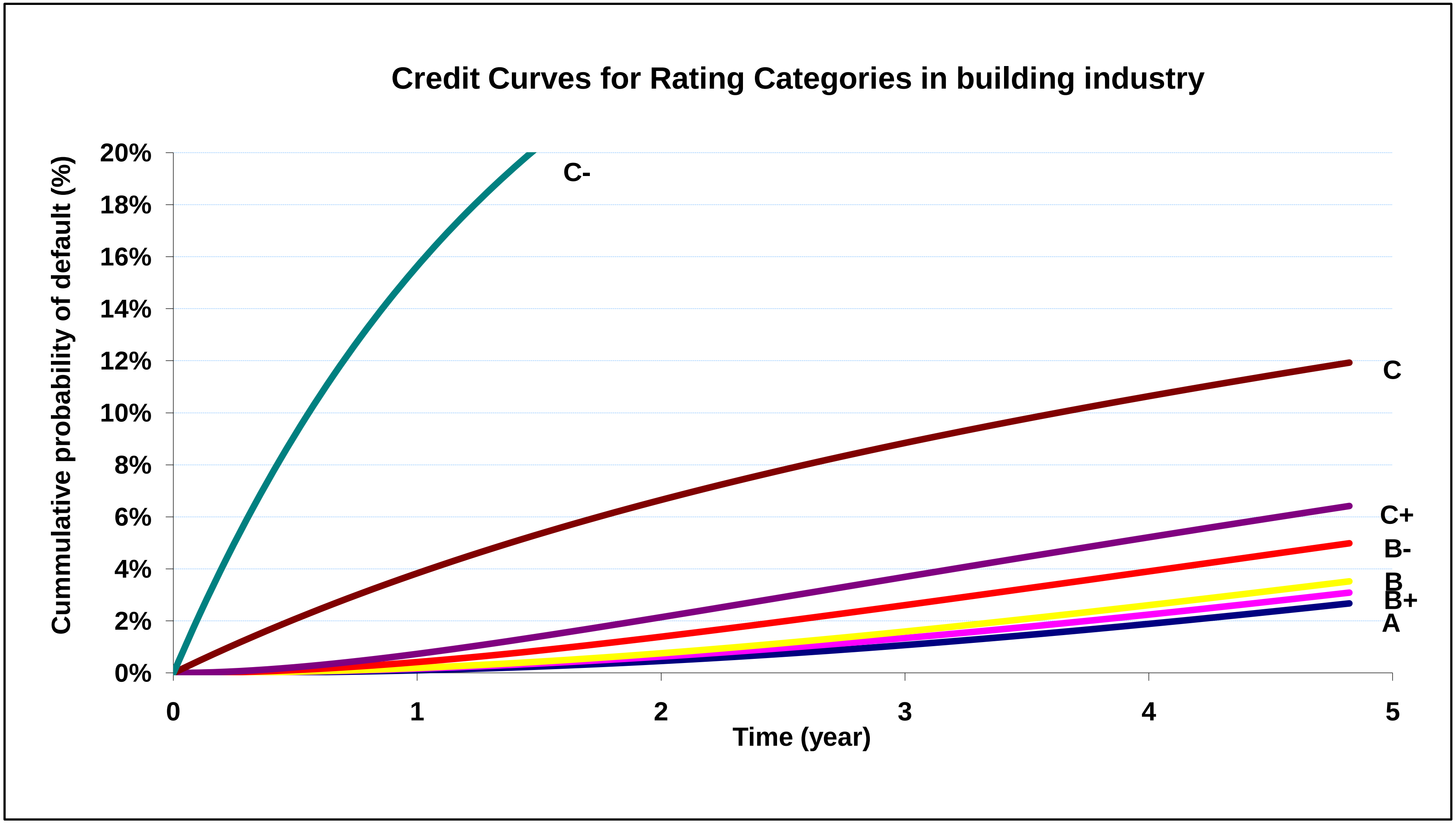
Credit curves

Transition matrix is used for computation the cumulative probability of default. For predefined time and industry branch is computed transition matrix and in the last column of the transition matrix is probability of the default for each rating category at beginning time, i.e.

$$Pst\left\{T_{Bankruptcy} \leq t \ / \ RatingLevel\right\}$$

	PB in specific time for building industry					
Rating From	1 Day	1 Week	1 Month	2 Months	6 Months	1 Year
A	0,000%	0,00%	0,00%	0,01%	0,022%	0,103%
B+	0,000%	0,00%	0,00%	0,01%	0,042%	0,166%
B	0,000%	0,00%	0,00%	0,01%	0,047%	0,195%
B-	0,000%	0,00%	0,00%	0,03%	0,112%	0,421%
C+	0,000%	0,00%	0,01%	0,06%	0,215%	0,735%
C	0,012%	0,09%	0,37%	1,07%	2,052%	3,835%
C-	0,060%	0,42%	1,74%	4,94%	9,075%	15,645%

Credit curves



Epilogue

Necessary conditions for successful implementation of the generator matrix:

- sufficiently accurate rating system, Gini index higher than 0,60
- sufficient amount of the rating migrations
- statistically stable occurrence of the ratings with convergence to the Gauss distributions

The generator matrix is useful for banks and financial institutions to estimate the expected loss of the bank loan and/or to estimate risk margin dependent on risk category and on time of exposition on credit risk

Literature:

[1] Credit Metrics™ – Technical Document J.P.Morgan & Co Incorporated, 1997,
[2] *Moody's* : Moody's Rating Migration and Credit Quality Correlation, 1920-1996, Moody's Investor Service, July 1997
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[4] *R.B.Israel, J. S. Rosenthal, and J.Z. Wei* : Finding Generators for Markov Chains via Empirical Transition Matrices. December 1999
[5] *Kreinin A. and M.Sidelnikova (2001)* : Regularization Algorithms for Transition Matrices ALGO RESEARCH QUARTELY, Vol. 4, NOS.1/2, MARCH/JUNE 2001

Thank you for your attention

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CZECH CREDIT BUREAU
A CRIF company