Convexity Adjustments for VaR

Peter Caspers
Eurobanking 2011

IKB Deutsche Industriebank

Generic VaR Computation

Input Parameters x

Probabilities of Default

Loss given Defaults

Correlation of Risk Factors

Volatilities of Risk Factors

↓ Model $f(\cdot)$

$VaR(\gamma, T) = f(x)$

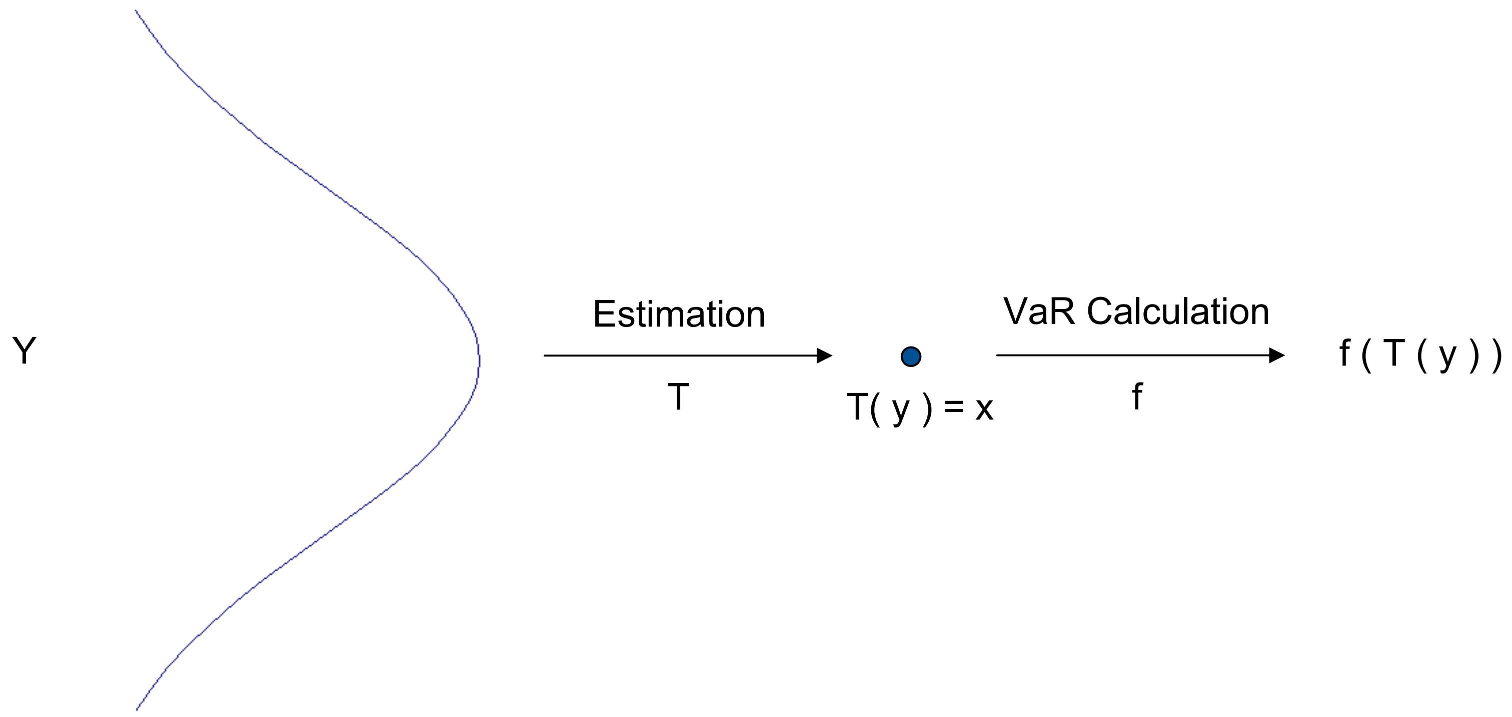
γ confidence level

T risk horizon

Deterministic input parameters

Input Parameters x are in general estimates of their true values.

In VaR calculation this is usually ignored:

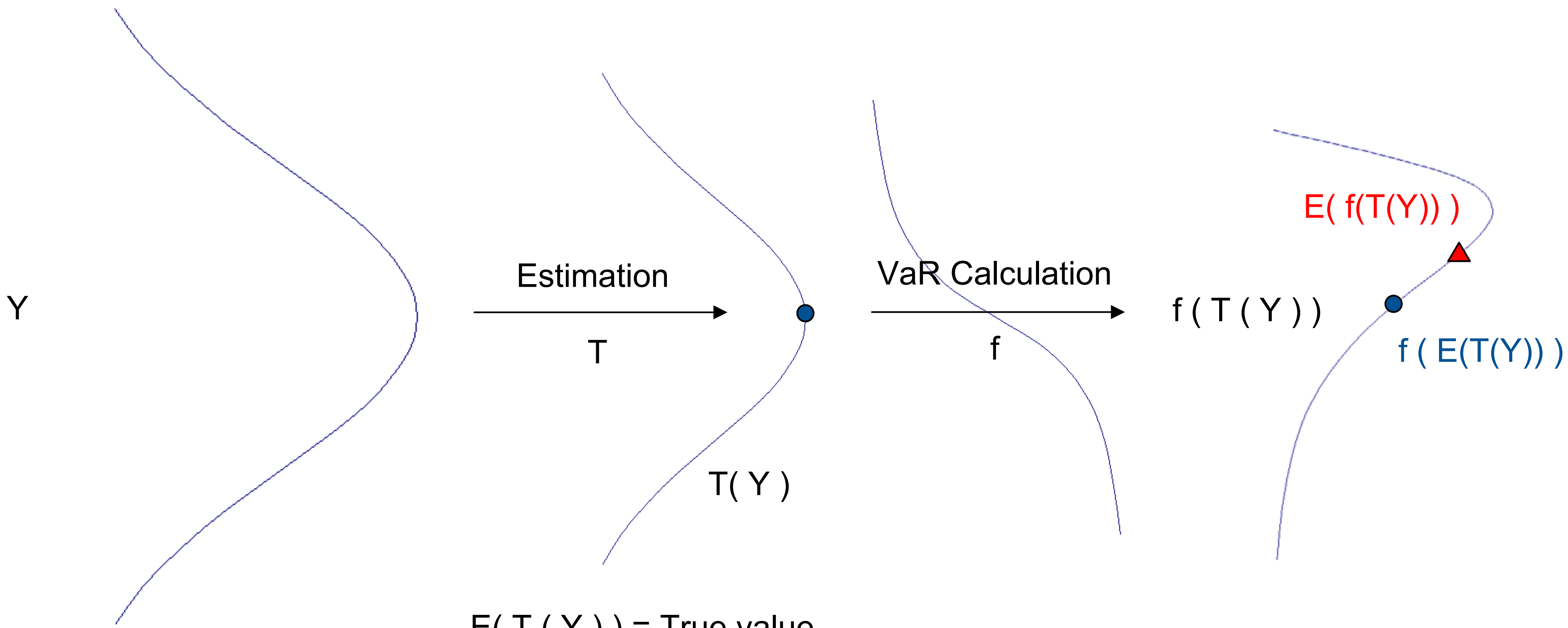


Random input parameters

Usual calculation: VaR = f (x)

But: Expected VaR = E (f(T(Y))) ≠ f (E(T(Y)))

due to non-linearity of f



E(T (Y)) = True value

If T is unbiased

Estimation risk

Let's call

f (E (T(Y))) - E (f (T(Y)))

the estimation risk. Then

True VaR = Usual VaR + Estimation Risk (in Expectation)

The estimation risk is a convexity adjustment to the usual risk computation, accounting for the uncertainty of parameter estimates.

The estimation risk is not zero as soon as

- f is non-linear and / or
- estimators of input parameters have non-zero variance

Both is usually the case.

Computing the convexity adjustment

The idea is to approximate f by a taylor expansion around the true value x and then write

$$E(f(T(Y))) \approx f(x) + \frac{1}{2}m_2 f^{(2)}(x) + \frac{1}{6}m_3 f^{(3)}(x) + \frac{1}{24}m_4 f^{(4)}(x) + \ldots$$

where m_2 is the Variance of T(Y).

The convexity adjustment or estimation risk is then expressed by

$$\begin{aligned} c =& c(m_1, m_2, \ldots) = f(x) - E(f(T(Y))) = \\ & -\left(\frac{1}{2}m_2 f^{(2)}(x) + \frac{1}{6}m_3 f^{(3)}(x) + \frac{1}{24}m_4 f^{(4)}(x) + \ldots\right) \end{aligned}$$

Note that in the actual computation of m_2 and $f^{(2)}(x)$ in practice the true value x has to be replaced by its estimate.

Specific example: ASFR model

The ASFR model computes a credit VaR taking an asset correlation R and a single probability of default PD as an input.

The latter parameter $x = PD$ is the one of interest, i.e. we want to consider the estimation risk of this parameter. R is fixed in our setup.

The model for the PD distribution is as follows: Fixing the number of obligors and R from above, we generate correlated defaults using a 1 factor gauss model. The number of defaults is then divided by the number of obligors resulting in a sample PD of our model distribution.

f, its second derivative and the moments of the PD estimator can be computed semi-analytically using a gauss hermite integration.

Computations in the ASFR

The VaR in the ASFR model is given by

$$\mathrm{VaR} = f(x) = \Phi(\alpha(x))$$

with x = PD and

$$\alpha(x) = (1-\rho^2)^{-0.5}(\Phi^{-1}(x) + \rho\Phi^{-1}(\gamma)) =: A\Phi^{-1}(x) + B$$

The derivatives of f are given by

$$\begin{aligned}
f' &= \phi\alpha' \\
f^{(2)} &= \phi'\alpha'^2 + \phi\alpha^{(2)} \\
f^{(3)} &= \phi^{(2)}\alpha'^3 + 3\phi'\alpha'\alpha^{(2)} + \phi\alpha^{(3)} \\
f^{(4)} &= \phi^{(3)}\alpha'^4 + 6\phi^{(2)}\alpha'^2\alpha^{(2)} + 3\alpha^{(2)}\phi' + 4\alpha^{(3)}\alpha'\phi' + \alpha^{(4)}\phi
\end{aligned}$$

Computations in the ASFR

where

$$\alpha' = \frac{A}{\phi}$$

$$\alpha^{(2)} = -\frac{A\phi'}{\phi^3}$$

$$\alpha^{(3)} = -\frac{A(\phi^{(2)}\phi - 3\phi'^2\phi)}{\phi^6}$$

$$\alpha^{(4)} = -\frac{A(\phi^7\phi^{(3)} + 2\phi^{(2)}\phi^6 - 12\phi'\phi^{(2)}\phi^6 + 15\phi^5\phi'^3)}{\phi^{12}}$$

and

$$\phi'(z) = -z\phi(z)$$

$$\phi^{(2)}(z) = -\phi(z) - z\phi'(z)$$

$$\phi^{(3)}(z) = -2\phi'(z) - z\phi^{(2)}(z)$$

Computations in the ASFR

The second moment of T is computed using the law of total variance

$$m_2 = V(T) = E(V(T|Z)) + V(E(T|Z))$$

where Z ~ N(0,1) is the systematic risk factor. The first term can be written as

$$\int_{-\infty}^{\infty} \frac{p(z)(1-p(z))}{n} \phi(z) dz$$

with p(Z) = PD | Z = conditional PD given Z. Using E(T|Z) = p(Z) the second term equals

$$V(p(Z)) = E(p(Z)^2) - E(p(Z))^2 = \int_{-\infty}^{\infty} p(z)^2 \phi(z) dz - \left(\int_{-\infty}^{\infty} p(z) \phi(z) dz \right)^2$$

Both integrals can be computed using a Gauss Hermite schema.

Specific example: Results

N	True f	$E(f)$	$E(f)+c(m_2^0)$	$E(f)+c(m_2)$	$E(f+c(m_2))$	MSE	$MSE_{c(m_2)}$
10	0.5355	0.3153	0.4282	0.5147	0.3834	0.1737	0.2007
50	0.5355	0.4211	0.4438	0.539	0.5112	0.0826	0.0833
100	0.5355	0.4373	0.4486	0.5439	0.5301	0.0661	0.062
500	0.5355	0.451	0.4533	0.5514	0.5384	0.054	0.0474
1000	0.5355	0.4523	0.4534	0.5509	0.5398	0.0526	0.046
5000	0.5355	0.4541	0.4543	0.5507	0.541	0.0516	0.0453
10000	0.5355	0.4543	0.4544	0.5517	0.541	0.0512	0.0449

Table 4: $PD = 0.08, \rho = 0.48$

The estimation risk is the higher the

- higher the PD
- higher the correlation
- smaller the number of obligors N

The correction – in terms of the bias – is quite effective, if N is not too small. For small N (e.g. N=10 above) the correction is too small.

Specific example: Results

N	True f	$E(f)$	$E(f)+c(m_2^0)$	$E(f)+c(m_2)$	$E(f+c(m_2))$	MSE	$MSE_{c(m_2)}$
10	0.5355	0.3153	0.4282	0.5147	0.3834	0.1737	0.2007
50	0.5355	0.4211	0.4438	0.539	0.5112	0.0826	0.0833
100	0.5355	0.4373	0.4486	0.5439	0.5301	0.0661	0.062
500	0.5355	0.451	0.4533	0.5514	0.5384	0.054	0.0474
1000	0.5355	0.4523	0.4534	0.5509	0.5398	0.0526	0.046
5000	0.5355	0.4541	0.4543	0.5507	0.541	0.0516	0.0453
10000	0.5355	0.4543	0.4544	0.5517	0.541	0.0512	0.0449

Table 4: $PD = 0.08, \rho = 0.48$

In terms of the MSE, the corrected estimator is not as convincing as when looking at the bias alone.

I.e. The improvement of getting closer to the true VaR in expectation is bought by a higher variance of the estimator.

Summary

A method for taking into account estimation risk defined as the difference between

- the expected VaR based on estimates of true input parameters and
- the VaR computed on basis of the true parameters themselves

is suggested.

The effect of the correction is demonstrated within the ASFR framework.

The bias can be reduced quite effectively in this example.

The variance however is increased by the correction, such that the MSE ist comparable to the uncorrected case.