Credit Spread Dynamics in First Passage Time Models

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1 Structural credit modelling

In structural credit modelling the credit quality is described by a stochastic *credit quality process* $X = (X_t)$ that drives the value of the firm. If the credit quality process falls below a certain *threshold* K default takes place; the default time τ is the *first passage time* of X below K:

$$\tau_K = \inf\{t \geq 0 : X_t < K\}. \tag{1}$$

The model has to be calibrated to credit spreads $s(0, T)$ (or default probabilities) given, e.g., by market quotes of CDS[1] or historical data.

Apart from the threshold K, which could be time dependent and a subject of calibration, the model for X should exhibit free parameters that can be calibrated.

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[1] Credit Default Swap

Desirably the model has more free parameters than needed for the mere calibration of spreads ⇒ *dynamics of future credit spreads* should be specifiable.

To make the calibration (and later the application of the model) feasible, a *closed form solution for the distribution of the passage time* τ_K is crucial.

2 First passage times for jump diffusions with Exponentially distributed jumps

The credit quality process X is a Lévy process:

$$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Z_k. \tag{2}$$

[Lipton (2002)] was one of the first to employ Lévy processes with exponentially distributed jumps Z_k to financial modelling.

For the *hyper-exponential jump diffusion* process (HEJD) the distribution of Z_k is a mixture of exponential distributions on the positive and negative half-axis, respectively. The density f_Z of Z_k is given by

$$f_Z(y) = \mathbf{1}_{y>0} \sum_{i=1}^{n^+} p^{+,i} \eta^{+,i} \exp\left(-\eta^{+,i} y\right) + \mathbf{1}_{y<0} \sum_{j=1}^{n^-} p^{-,j} \eta^{-,j} \exp\left(\eta^{-,j} y\right).$$

A special case of a HEJD is the double exponential jump diffusion model studied by [Kou(2002)], where $n^+ = n^- = 1$.

As shown by [Crosby, Le Saux and Mijatović(2009)], a large class of Lévy processes can be approximated by HEJDs.

HEJDs afford a high degree of analytical tractability since many quantities of interest are known, in particular, the Laplace transform of passage times.

The moment generating function $\mathbb{E}\exp(uX_t) = \exp(G(u)t)$ of X_t has the exponent:

$$G(u) = u\mu + \frac{1}{2}u^2\sigma^2 + \lambda \sum_{i=1}^{n^+} p^{+,i}\left(\frac{\eta^{+,i}}{\eta^{+,i}-u} - 1\right) + \lambda \sum_{j=1}^{n^-} p^{-,j}\left(\frac{\eta^{-,j}}{\eta^{-,j}+u} - 1\right). \tag{3}$$

For every $\alpha > 0$, the equation

$$G(u) = \alpha \tag{4}$$

has $n^+ + 1$ real positive roots $\beta^+_{1,\alpha}, \ldots, \beta^+_{n^++1,\alpha}$ and $n^- + 1$ real negative roots $-\beta^-_{1,\alpha}, \ldots, -\beta^-_{n^-+1,\alpha}$, [Mordecki(2002)].

[Lipton (2002)] calculated the Laplace transform of the distribution of the passage time τ_K as

$$\mathcal{L}(K,\alpha) = \mathbb{E}\left[\exp(-\alpha\,\tau_K)\right] = \sum_{j=1}^{n^-+1} A_j^- \exp\left(\beta_{j,\alpha}^-\,K\right), \tag{5}$$

where

$$A_j^- = \frac{\prod_{v=1}^{n^-}\left(1-\frac{\beta_{j,\alpha}^-}{\eta_v^-}\right)}{\prod_{v=1,v\neq j}^{n^-+1}\left(1-\frac{\beta_{j,\alpha}^-}{\beta_{v,\alpha}^-}\right)}.$$

3 Credit Modelling for known Laplace transform

The default time is modelled by τ_K. Pricing (calibrating to) credit default swaps (CDS) we have to consider the *risky present value of a basis point* $PVBP$ and the *present value of default protection* D with maturity T:

$$PVBP(T) = \mathbb{E}\left[\int_0^T \exp(-ru)\mathbf{1}_{\tau_K > u}\,\mathrm{d}u\right], \qquad D(T) = \mathbb{E}\left[\exp(-r\tau_K)\mathbf{1}_{\tau_K \le T}\right].$$

If the CDS spread s is *paid continuously*, the value $V_s(T)$ of a CDS is

$$V_s(T) = s\,PVBP(T) - (1 - R)\,D(T).$$

The fair CDS spread is $s(0,T) = \frac{(1-R)D(T)}{PVBP(T)}$.

Proposition 1. *The Laplace transforms of the value $V_s(T)$ of the CDS, of the risky present value of a basis point $PVBP(T)$ and of $D(T)$ are*

$$\begin{aligned} \mathcal{L}(V_s)(\alpha) &= s \cdot \frac{1-\mathcal{L}(K,\alpha+r)}{\alpha(\alpha+r)} - (1-R)\cdot\frac{\mathcal{L}(K,\alpha+r)}{\alpha}, \qquad (6) \\ \mathcal{L}(\mathit{PVBP})(\alpha) &= \frac{1-\mathcal{L}(K,\alpha+r)}{\alpha(\alpha+r)}, \\ \mathcal{L}(D)(\alpha) &= \frac{\mathcal{L}(K,\alpha+r)}{\alpha}, \end{aligned}$$

where $\mathcal{L}$ is the Laplace transform of τ_K as given by (5).

Inverting the Laplace transforms of D and of $PVBP$, we can determine the quantities $D(T)$ and $PVBP(T)$ and calculate the fair spread $s(0,T)$.

3.1 Fast calibration

Calibrating amounts to backing out the parameters μ, σ, λ, $\eta^{+,i}$, $i = 1, \ldots, n^+$, $\eta^{-,j}$, $j = 1, \ldots, n^-$, etc., from market quoted CDS spreads $s_{\text{market}}(0, T)$. The calibrated model has to satisfy $V_{s_{\text{market}}(0,T)}(T) = 0$. By (6), calibrating the model parameters numerically actually requires just $one\ Laplace\ inversion$ per calibration iteration step.

The number of model parameters to be calibrated is considerable. For the mere calibration problem a spectrally negative Kou model $(n^+ = 0,\, n^- = 1)$[2] already achieves convincing fits to typical CDS-curves. The calibration error attains close to minimum values for many different combinations of parameters[3] σ, λ, η^-:

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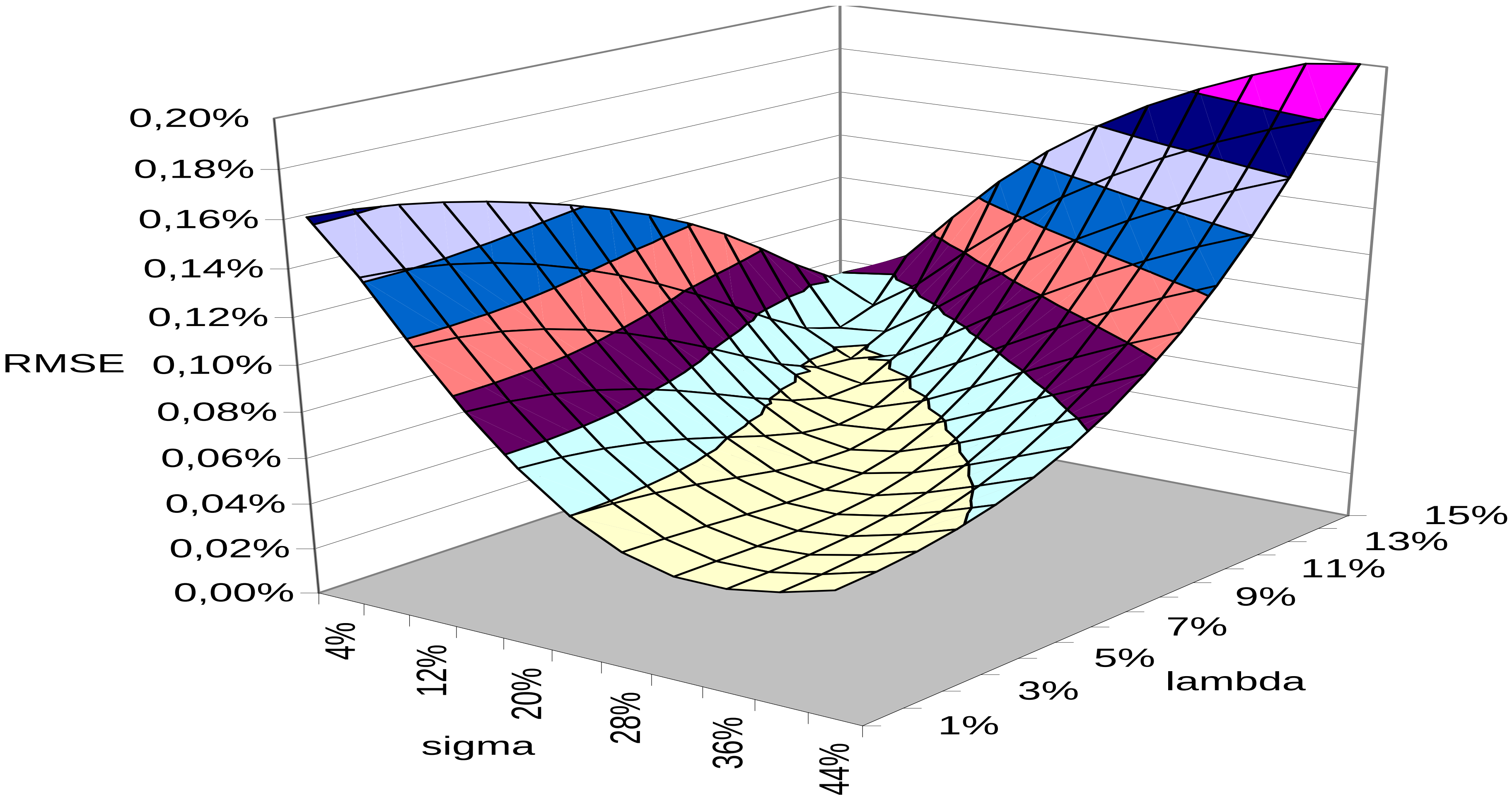
[2] The three roots of the equation $G(u) = \alpha$, are then known analytically.

[3] We set $K = -1$ and determine μ from a martingale condition.


RMSE
0,20%
0,18%
0,16%
0,14%
0,12%
0,10%
0,08%
0,06%
0,04%
0,02%
0,00%
sigma
4%
12%
20%
28%
36%
44%
lambda
1%
3%
5%
7%
9%
11%
13%
15%

3.2 Credit dynamics

Calibration retains free parameters to specify the dynamics of credit spreads. The time t value of the risky present value of a basis point $PVBP(t, T)$ and the value of a payment in default $D(t,T)$ are,

$$PVBP(t, T) = \mathbb{E}\left(\int_0^T \exp(-ru)\mathbf{1}_{\tau_K > t+u}\,\mathrm{d}u \middle| \mathcal{F}_t\right)$$

$$D(t,T) = \mathbb{E}\left(\exp(-r(\tau_K - t))\mathbf{1}_{t < \tau_K \leq t+T} \middle| \mathcal{F}_t\right).$$

The fair CDS spread at time t is $s(t,T) = \frac{(1-R)D(t,T)}{PVBP(t,T)}\mathbf{1}_{\tau_K > t}$. By the Markov property of (X_t), the quantities $PVBP(t, T)$, $D(t,T)$, and $s(t,T)$ are functions of X_t.

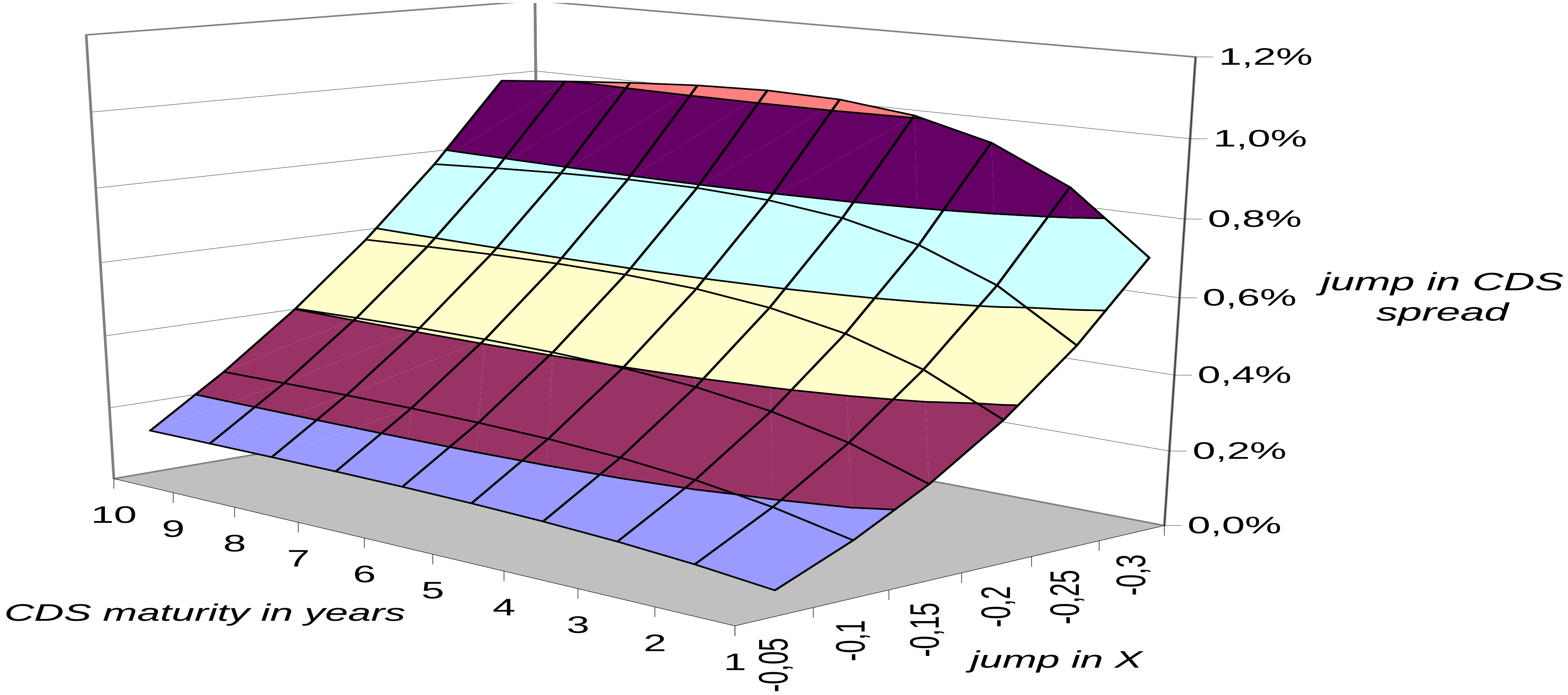
Analogously to Proposition 1, the Laplace transform of $PVBP(t,T)$, $D(t,T)$, over T can be made explicit. We investigate the dynamics of the spread $s(t,T) = F^s(T, X_t)$, $t \geq 0$. By Ito's formula

$$\begin{aligned} s(t,T) &= s(0,T) + \int_0^t \left[F^s_x(T, X_s)\mu + F^s_{xx}(T, X_s)\frac{\sigma^2}{2} \right] \mathrm{d}s \\ &\quad + \int_0^t F^s_x(T, X_s)\sigma \, \mathrm{d}W_s + \sum_{0<s\leq t} \left[F^s(T, X_{s-} + \Delta X_s) - F^s(T, X_{s-}) \right]. \end{aligned} \tag{7}$$

Lemma 1. [Spread jumps]

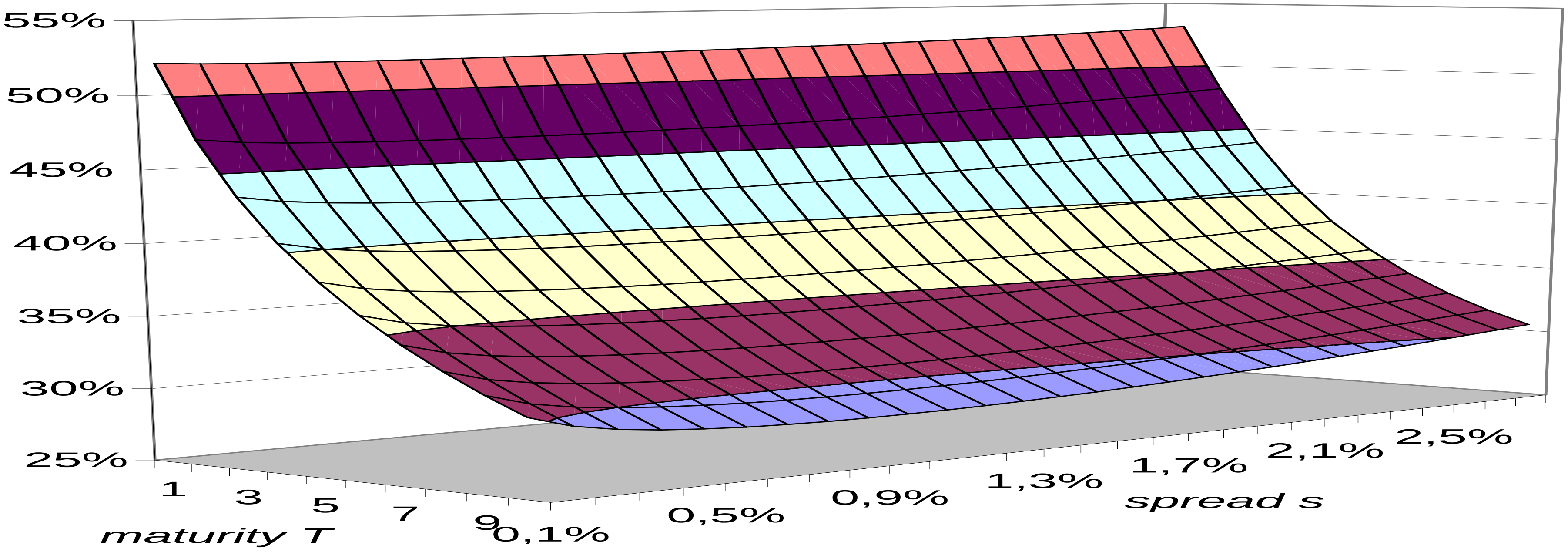
$$\begin{aligned} \Delta s(t,T) > 0 &\iff \Delta X_t < 0, \\ \Delta s(t,T) < 0 &\iff \Delta X_t > 0. \end{aligned}$$

For the calibration $\sigma = 0.12$, $\eta^- = 5.0$, and $\lambda = 0.3$ to the curve $s(0,1) = 40.9$bp, $s(0,3) = 65.3$bp, $s(0,5) = 80.5$bp, $s(0,7) = 81.9$bp, $s(0,10) = 82.9$bp:



Lemma 2. [Local Spread Volatility] *The diffusion part of* $s(t,T)$ *has local volatility*

$$v(T,s) = \frac{\sigma F^s_x(T,x)}{F^s(T,x)}, \text{ where } F^s(t,x) = s. \tag{8}$$



4 Application: pricing gap risk

Gap risk is the risk originating from the occurrence of jumps in credit spreads.

Leveraged Credit Linked Note:
Investment of 1 USD is deposited in risk free account earning interest r. Investors sells protection in a CDS with maturity T on a leveraged notional amount of $k > 1$ USD, earning a spread income of $ks(0, T)$. In case of default and $k(1 - R) > 1$, the investment amount of 1 USD is not sufficient to cover the losses from the CDS ⇒ unwind the position before default at trigger time

$$S = \inf\{t \in [0, T] : V_t^k \leq -L\} \text{ with } 0 < L \leq 1.$$

loss	investor	issuer
$V_S^k \geq -1$	V_S^k	0
$V_S^k < -1$	$\max(V_S^k, -1)$	$V_S^k + 1$
premium	$\tilde{k}s(0,T)$	$(k-\tilde{k})s(0,T)$

$$0 = \mathbb{E}_{\mathbb{Q}}\left((k-\tilde{k})s(0,T)\int_0^{T\wedge S} \mathbf{e}^{-ru}\mathrm{d}u + \min(1+V_S^k, 0)\mathbf{e}^{-rS}\mathbf{1}_{S\leq T}\right). \qquad (9)$$

credit quality process X	$\tilde{k}$
pure diffusion, $\lambda = 0$	k
pure jump $\sigma = 0, \eta^-$ small	$\frac{1}{1-R}$

Given a CDS spread curve of

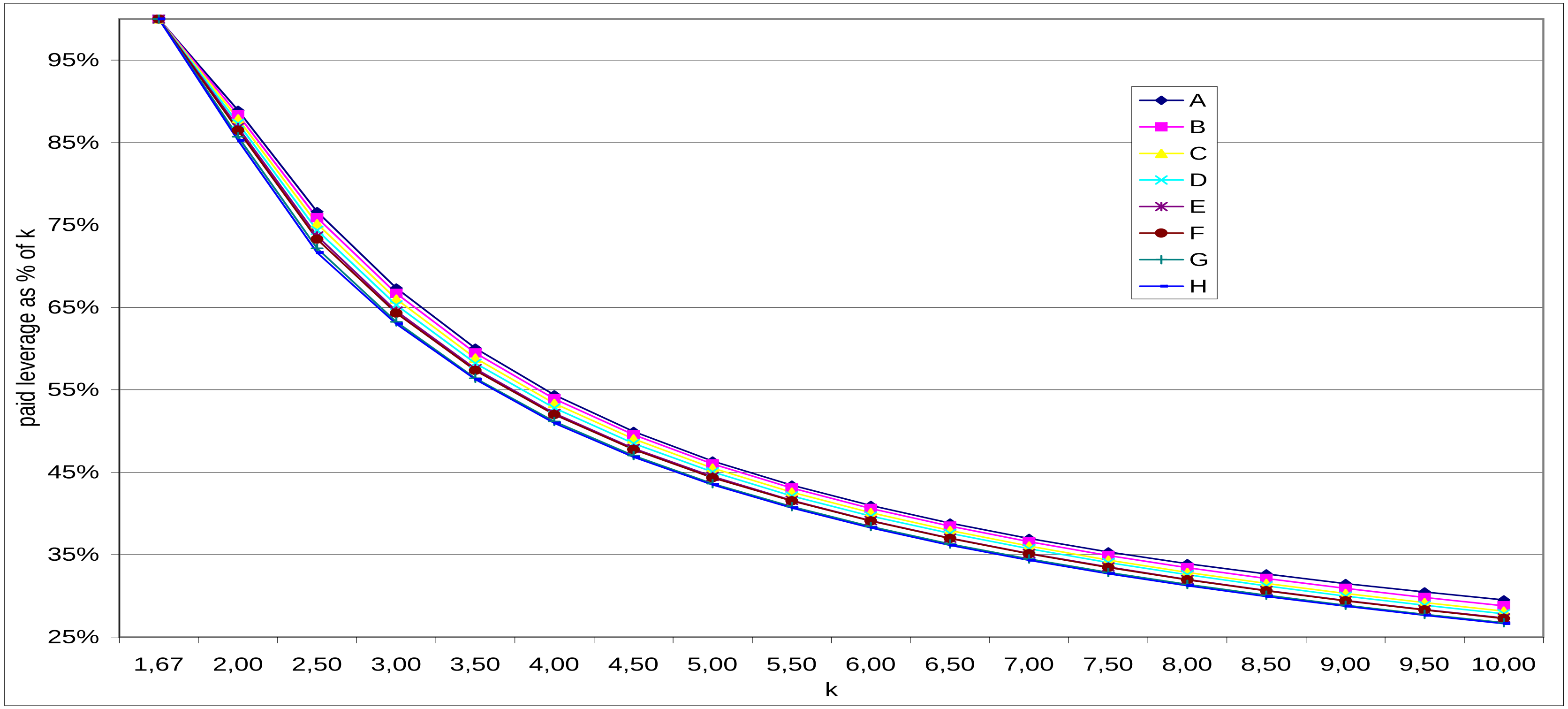
CDS maturity	1Y	3Y	5Y	7Y	10Y
CDS spread	0,80%	1,00%	1,10%	1,20%	1,25%

the following parameter constellations all provide an excellent fit :

setup	A	B	C	D	E	F	G	H
σ	12%	10%	9%	8%	7%	6%	4%	1%
λ	14%	20%	22%	24%	26%	28%	30%	32%
η^-	2,595	3,000	3,088	3,171	3,250	3,328	3,373	3,428

We assumed a risk free rate $r = 3\%$ and recovery rate $R = 40\%$.

Consider a leveraged note with maturity $T = 10$ years and set the trigger level to $-L = -0.5$. For a range of leverages k we calculated the fair leverage factor $\tilde{k}$ to be paid to the investor of the note.



Surprisingly, we face less model risk as expected at first thought. These are good news for pricing leveraged credit linked notes in the framework of our model. However, one can expect that the different model setups lead to different hedging strategies.

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